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PRESSURES ON AMERICAN MATHEMATICS TEACHERS[1]

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American teachers of mathematics are at present subject to pressure from four directions, corresponding to four aspects of the actual situation.

(1) *The prevalence of rote learning.* Not only in schools but often in college teaching, mathematics is treated as a series of rules that must be memorized.

(2) *The comparison with Europe.* In many European countries (including Russia of course for headline purposes) pupils meet subjects about two years earlier than American children do; the flexibility of European education often allows particularly able children to be four years ahead of their American counterparts.

(3) *Modern mathematics.* "The present syllabus in our high schools corresponds almost exactly to what was known in the year 1640."

(4) *Modern Technology.* Automation, electronic computers, social sciences, etc. require new kinds of mathematics.

It is natural that different people see different parts of the problem. The research mathematician is particularly aware that no twentieth century mathematics is dreamed of by most teachers. The technologist is concerned with the industrial applications. The college professor of physics or engineering wants to teach mechanics to college freshmen and would rejoice if high schools gave even a seventeenth century intuitive understanding of calculus. The good teacher in school or college regrets the prevalence of rote learning and its destruction of initiative and curiosity. The school teacher may observe how bored the best students are by the long dragging out of elementary arithmetic, so that some of the ablest turn their backs on mathematics as a dull subject.

Each and every one of these viewpoints reflects one aspect of the truth. The danger, in this age of specialization and intellectual atomization, is that each aspect tends to become a separate creed, instead of an element in our awareness of the whole situation. Any one of these partial views can defeat its own ends.

For example, most research workers in pure mathematics are moved mainly by the inherent beauty and interest of their work. This is entirely justifiable for a research specialist. I have heard a distinguished mathematician claim that none of the really new mathematics had found any practical application whatever; he defended it as "an art form". It would be a mistake to send a man, holding this view, to plead with Congress for increased expenditure on mathematical education, or to persuade a keen young scientist in High School that it was worth while to take several mathematics courses. If mathematics were simply what *such*

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a pure mathematician claims, our present acceptance of public funds would be a gigantic swindle. Fortunately, it is not.

To form a rational policy it would be necessary to have (i) a good knowledge of all branches of mathematics, both of this and of previous centuries, (ii) a knowledge of its scientific and industrial applications, (iii) an estimate of the number of people engaged in mathematical research, in scientific research using considerable mathematics, in occupations calling for some mathematical knowledge, (iv) sufficient experience of teaching students at all ages and ability levels. No one claims such encyclopedic knowledge. But this theoretical ideal is one that should constantly be borne in mind when making, or judging, proposals for mathematical education. One can distinguish a serious program from an expression of individual or sectional prejudice by the extent to which it takes account of *all* of these factors.

It would at present be very valuable if we could produce a number of studies of the scientific uses of pure mathematics. An applied mathematician frequently finds himself in this position; he has a definite problem that he wants to solve; it seems possible that a certain branch of pure mathematics will throw light on it; but this branch may take several months to learn. It would be most useful to know (a) whether this branch will in fact contribute anything to the immediate problem, (b) how deeply one needs to go into that branch for the particular purpose in mind.

With the co-operation of mathematicians and scientists interested in this aspect of things, it would not be too difficult to produce such a survey. The ingredients probably exist already, scattered about the place. For example, Hilbert space has well known applications in quantum theory. Halmos' little book, *Introduction to Hilbert Space*, opens with a most useful section, "Prerequisites and Notation" from which one can derive a fair idea of the mathematical subjects involved and the extent (in general, slight) to which knowledge of them is required.

Having such a survey, we could make a case for "modern mathematics" on the basis of reason, instead of a mixture of guesswork, aesthetic considerations, and the desire to be in the fashion.

A GUIDING PRINCIPLE

We are faced with this tremendous task of catching up with three centuries of mathematical discovery through an educational system that barely manages to teach arithmetic efficiently. Not through the fault of individual teachers, but through administrative policy, no margin of safety has been left, no provision for growth. It would be vastly different if we knew that every teacher of arithmetic had a good understanding of algebra, and every teacher of trigonometry had calculus in reserve. Then an orderly change of syllabus could easily be arranged. As things are, we know that many things which ought to be done will not be done. How shall we decide what points to abandon, and what to cling to as the key to the whole situation?

As a hypothesis I would urge that *the elimination of rote learning* is that key. The whole emphasis of teaching (at all levels) should be on insight. "Can you see clearly what the problem is?" "What do you think would be a reasonable way of attacking it?" "Here is a statement; how would you test for yourself whether it is true or not?" Both in mathematics and science we should encourage the student to rely upon his own observation, his own experience, his own common sense. So, and only so, does knowledge rest securely in the mind.

There is nothing original in this view. Most able people in industrial and scientific institutions, when asked what they like an entrant to know, answer, "Above all, we want him or her to know how to think." This surely agrees with our own experience as working scientists or mathematicians. If we have to learn a new subject, we can do so fairly easily, for we are used to attacking new problems. The rote-learning student, on the other hand, is helpless in the face of a new situation.

But genuine thinking implies continuity. New discoveries are suggested by old knowledge. When this continuity is broken, mystification results. Thus rote learning in arithmetic appears when a child fails to associate the symbol $2 + 3$ with the physical experience of putting two and three stones together. Rote learning appears in algebra when children fail to test algebraic statements against their knowledge (if any) of arithmetic. Rote learning appears in modern algebra when the student is insufficiently familiar with elementary mathematics.

Historically, this continuity is very evident. The broad generalizations arose on the basis of a wealth of particular results. For example, Steinitz mentions in the introduction to his theory of fields that he was led, by Hensel's development of p-adic fields, to enquire into the nature of *all* possible fields. When a particular result seems valuable or interesting, a mathematician naturally tries to generalize it. When a large number of such generalizations have proved fruitful, the belief spreads that generalization is the thing to do. This is essentially the mood of the present century. It was not the mood of 1640, and it requires careful justification to the high school student or teacher who is not familiar with the recent history of mathematics.

In our desire to be modern, we need to exercise extreme care if we are not simply to produce yet one more breach of continuity and a new fashion in rote learning. Few mathematicians realize the enormous gulf that separates them from the majority of school teachers. This has led to some fiascos, when mathematicians have tried to communicate with teachers. The mathematician concludes that the teachers are stupid; the teachers that the mathematician is a poor lecturer. Neither need be true. The teachers are quite capable of understanding modern mathematics, if brought to it by sufficiently gradual steps; the mathematician may be illuminating to a student familiar with the current mathematical terminology. The fact simply is that past methods of mathematical education have produced a society split clean down the middle.

This fact was strongly brought home to me when I found that an introduction to modern algebra, which I had always regarded as a model of lucidity, was completely unreadable by a highly selected group of able high school teachers of mathematics. With an unselected group, things are even more acute. In a summer school on modern algebra, one cannot assume knowledge of the Remainder Theorem in elementary algebra; there may be students who draw fine philosophical distinctions between numbers and numerals, but cannot solve a quadratic equation, and are somewhat shaky on linear equations.

With such a lack of concrete background, anything except rote learning of modern mathematics is unlikely. A very carefully devised progression from the concrete to the abstract is required to bridge the gap.[3]

The view tends to be held that an accurate verbal definition (e.g., of a function) conveys understanding. This thesis seems highly questionable. Without experience, words convey nothing. It can of course be extremely valuable for someone who has met and become familiar with a wide variety of functions, *and seen the specific mathematical difficulties to which narrower definitions lead,* to extract from this experience the essential idea of a table, a mapping, ordered pairs, or what you will.

Premature rigor is another cause of difficulty. Marshall Stone's dictum, "The basic courses in mathematics should not dwell unduly or prematurely on mathematical and logical niceties" is eminently sound.[4]

THE NONMATHEMATICAL USERS OF MATHEMATICS

Stone's dictum may appear alarming to those who think that nothing should be told the student unless it is preceded by a rigorous proof. Consider, however, how much poorer mathematics would be if unproved statements had always been excluded from it. Seventeenth century mathematics, the historical origin of nearly all later work, would disappear completely. Fermat's Last Theorem, Goldbach's conjecture, the Riemann Hypotheses, all of Riemann's function theory based on Dirichlet's Principle—these would never have been published. Mathematics grows by the gradual clarification of ideas, by the gradual transition from evidence to proof. There is an immense loss if the teacher is forbidden to say to the student, "You can picture this idea, and see intuitively that it is reasonable. It has in fact been proved." This approach is particularly important for those who are users, rather than makers, of mathematics.

In speaking of users of mathematics, I do not wish to imply that education should be geared exclusively to technology. If, for example, we teach something because it arouses the interest of students and encourages intellectual curiosity—that is indeed admirable.

But education clearly should prepare people to cope, efficiently and without anxiety, with those problems that they actually meet in the course of their daily work.

In 1945–1947, I made a survey of the industrial uses of mathematics in Leicester, England, a city with a wide variety of industries, though not, of

[3] I hope to deal with this matter in greater detail some time, in a teachers' journal.

[4] The Mathematics Student, XXIV (1956), p. 34.

course, with the automation and other developments of the last decade. There were a few key individuals who required an extremely high degree of mathematical ability. There were many more who required mathematics mainly as a *language*: the doctor who used statistics for medical research, the research chemist who needed to read papers in mathematical physics, the engineering student who needed to follow a textbook using mathematical symbols. Occasionally such people needed mathematical rigour; the textbook might use a fallacious argument. More often they needed the power to visualize mathematical symbols in physical terms. If I denoted the current in the primary of a transformer, the student needed to see just what was meant when dI/dt came into the argument.[5]

The old survives alongside the new. There are still many doctors, scientists, and engineers for whom the ability to read traditional mathematics is still the decisive requirement.

The old should certainly not be pushed out of the curriculum. But there are certain new things that could well come in. The Boolean algebra of electrical switching could be taught extremely early. It would have a topical appeal to students. It is extremely concrete. It requires practically no technical background. It is easier to teach than ordinary algebra, since it does not require familiarity with arithmetic. Being a new subject, it might make a breach in the tradition of rote learning. A text could be written in which the pupil discovered the subject entirely for himself, and would be able to compare the structure of this algebra with that of the usual school algebra.

THE MECHANISM OF CHANGE

Any change in syllabus, particularly so radical a change as would allow the ablest students here to advance abreast of their European counterparts (i.e., traditional algebra, geometry, trigonometry, and some calculus by the fourteenth birthday), presents apparently insoluble problems in the retraining of teachers, the reorganisation of training colleges, and the rewriting of college entrance regulations.

However such a revolution has already occurred twice in this century—in the automobile and radio industries. *Neither of these grew through changes in school syllabus*. Both were made possible by boys studying independently, making cars, making radios in their own homes.

The present revolution in mathematics and technology can most easily be brought about along the same lines. Our main task is to produce literature that will be intelligible and attractive to students *from Grade 5 upwards*. For readers aged ten to fourteen, the literature should mainly be concerned with traditional

[5] Any teacher of college calculus who succeeds in conveying this understanding can pride himself on being exceptional. As a correspondent wrote to me, "As you well know, in an Engineering College, mathematics are thrown at you in chunks, as you might throw a man a wrench.

Theirs not to reason why
Theirs not to make reply
Theirs but to use it or die."

mathematics and mechanics, though showing of course its relevance to contemporary developments and including any modern development that is sufficiently simple—such as switching theory.

Anyone who cares to walk into any American grade school can verify for himself that there are plenty of students at least as good as European students, and perfectly capable of digesting a European syllabus. The limiting factor is the capacity of adults to provide, not of youth to absorb.

If we can, in some measure, bring about such a change now, among the youth, we shall have the teachers and the scientists we need fifteen or twenty years hence. For our more urgent needs, we can follow a similar approach with somewhat older students.